RAMSEY EQUIVALENCE OF K_n AND $K_n + K_{n-1}$

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Abstract. We prove that, for $n \geq 4$, the graphs K_n and $K_n + K_{n-1}$ are Ramsey equivalent. That is, if G is such that any red-blue colouring of its edges creates a monochromatic K_n then it must also possess a monochromatic $K_n + K_{n-1}$. This resolves a conjecture of Szabó, Zumstein, and Zürcher [10].

A finite graph G is Ramsey for another finite graph H, written $G \to H$, if there is a monochromatic copy of H in every two-colouring of the edges of G. We say that H_1 and H_2 are Ramsey equivalent, written $H_1 \sim_R H_2$, if, for any graph G, we have $G \to H_1$ if and only if $G \to H_2$.

The concept of Ramsey equivalence was first introduced by Szabó, Zumstein, and Zürcher [10]. A fundamental question to ask is which graphs are Ramsey equivalent to the complete graph K_n. It follows from a theorem of Folkman [5] that if a graph H is Ramsey equivalent to K_n, then $\omega(H) = n$, where $\omega(H)$ denotes the size of the largest complete subgraph of H.

In a recent paper, Fox, Grinshpun, Person, Szabó, and the second author [6] showed that K_n is *not* Ramsey equivalent to any connected graph containing K_n. Furthermore, it is easily seen that K_n is not Ramsey equivalent to $K_n + K_n$[1], see e.g. [10]. It follows that if $K_n \sim_R H$ then H is of the form $K_n + H'$ where $\omega(H') < n$.

It was shown in [10] that $K_n \sim_R K_n + K_{n-2}$. However, the question of whether K_n is Ramsey equivalent to $K_n + K_{n-1}$ was left open. It is easily checked that K_3 is not Ramsey equivalent to $K_3 + K_2$, since $K_6 \to K_3$ but $K_6 \nrightarrow K_3 + K_2$. In [10] it is conjectured that this is an aberration, and that $K_n \sim_R K_n + K_{n-1}$ for large enough n, a conjecture repeated in [6]. In this paper, we prove this conjecture.

Theorem 1. *For any $n \geq 4$*

$$K_n \sim_R K_n + K_{n-1}.$$

It is shown in [6] that this is best possible, in the sense that K_n is not Ramsey equivalent to $K_n + 2K_{n-1}$.

Our methods are combinatorial and explicit, and the idea is the following: suppose we have a graph G which is Ramsey for K_n, and yet has been coloured so as to avoid a monochromatic $K_n + K_{n-1}$. We will then attempt, by giving an explicit recolouring of some edges, to give a colouring which no longer possesses a monochromatic K_n, which contradicts the Ramsey property of G.

The first author was supported by the Heilbronn Institute for Mathematical Research. The second author was supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007- 2013)/ERC grant agreement no. 259385.

[1]Throughout the paper, we denote by $H_1 + H_2$ the graph composed of two vertex-disjoint copies of H_1 and H_2. More generally, we denote by $H_1 + tH_2$ the graph that consists of a copy of H_1 and t pairwise vertex-disjoint copies of H_2.

Sadly, this is not quite possible directly, and instead we will build up our proof in stages: in each lemma, we will show that either a colouring of G must have a monochromatic $K_n + K_{n-1}$, or if not we can deduce some further structural information about the colouring of G, which will help us in the following lemmas. Eventually, we will have accumulated enough information about our supposed counterexample so that it collapses under the weight of contradiction into non-existence, which proves Theorem 1.

As mentioned above, the clique on six vertices is an unfortunate obstruction which prevents the Ramsey equivalence of K_3 and $K_3 + K_2$. Interestingly, Bodkin and Szabó [2] have shown that, essentially, this is the *only* such obstruction.

Theorem 2 ([2])**.** *If $G \to K_3$ and $G \nrightarrow K_3 + K_2$ then $K_6 \subseteq G$.*

In Section 3, we give an alternative proof of this theorem, using similar techniques to those developed for the proof of Theorem 1.

Notation. All graphs are simple and finite. As a convenient abuse of notation, we write G both for a graph and for its set of vertices. We write $E(G)$ for the set of edges of G.

Structure of the paper In Section 1 we prove a Ramsey stability lemma, crucial for the proof of Theorem 1, but which is also of independent interest. In Section 2 we give the proof of Theorem 1. In Section 3 we give the proof of Theorem 2. Finally, we conclude by giving further discussion of Ramsey equivalence, including a discussion of some still-open conjectures in this field, and adding some more.

1. Ramsey stability

We first prove a lemma which may be of independent interest; we refer to it as a Ramsey stability result, since it states that if a graph G is Ramsey for the clique K_n then we can remove any small number of vertices and the remaining graph will still possess a Ramsey property almost as strong as the original.

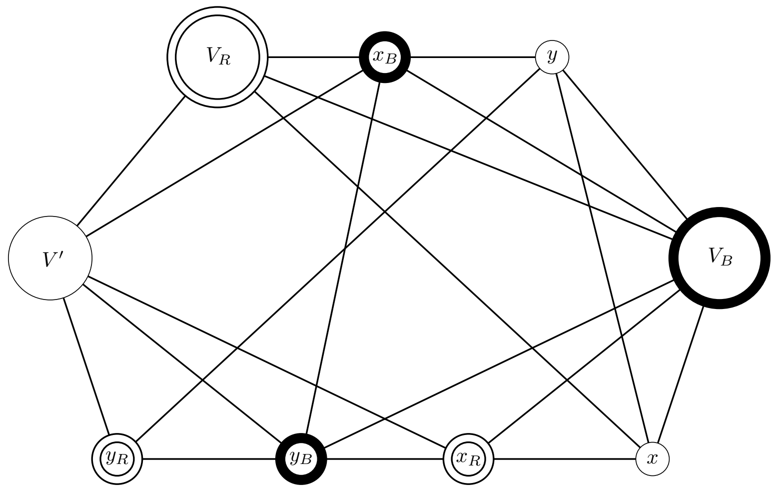
Lemma 1. *Let $n \geq 4$ and $G \to K_n$. Let $V \subset G$ with $2n \leq |V| \leq 3n-3$ and $V_0 \subset V$ be any set with $|V_0| \leq 2n-2$. Finally, let x and y be any vertices from $V \setminus V_0$.*

Then, in any colouring of the edges of G, there exists a monochromatic copy of K_{n-1} in $G \setminus V_0$, say with vertex set W, such that either $W \cap V = \{x\}$, or $W \cap V = \{y\}$, or $x, y \notin W \cap V$.

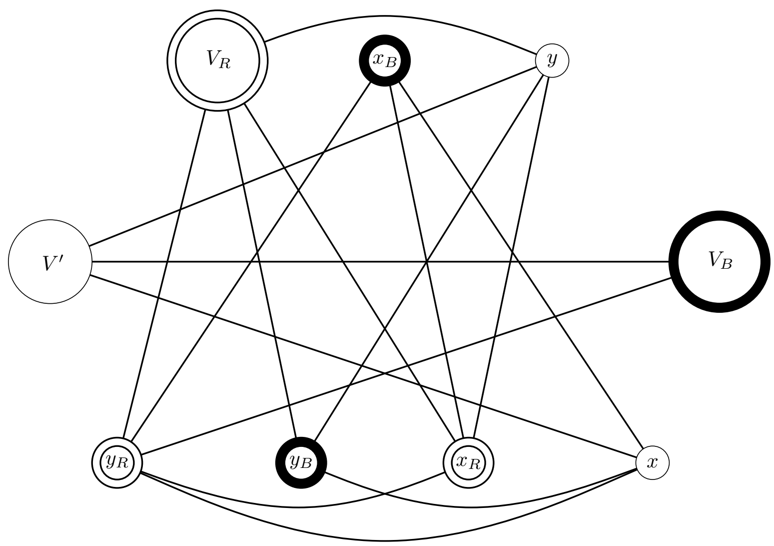
Proof. Without loss of generality, we may suppose that $|V| = 3n-3$ and $|V_0| = 2n-2$. We arbitrarily divide V_0 into two sets of $n-3$ vertices each, say V_R and V_B, and four single vertices, x_R, y_R, x_B, y_B. For brevity, we let $V' = V \setminus (V_0 \cup \{x, y\})$. To define a recolouring of the edges incident to V, let us define an auxiliary graph G_R with vertex set

$$V(G_R) = \{V_R, V_B, V', \{x\}, \{y\}, \{x_R\}, \{y_R\}, \{x_B\}, \{y_B\}\}.$$

Instead of giving an incomprehensible list of edges, we refer the reader to Figure 1 (A) for the definition of G_R. Let G_B be the complement of G_R, depicted in Figure 1 (B). We now recolour the edges incident to V as follows. If $u_1 \in U_1 \in G_R$ and $u_2 \in U_2 \in G_R$ such that $U_1 \neq U_2$, then colour the edge u_1u_2 red if $U_1U_2 \in E(G_R)$, and colour the edge u_1u_2 blue otherwise. Furthermore, colour all edges in $E(V_B)$ red, and all edges in $E(V_R)$ blue. The edges in $E(V')$ retain their original colouring.



(A) The (red) edges of G_R



(B) The (blue) edges of G_B

FIGURE 1. The recolouring of V for Lemma 1. A black ring around a vertex class indicates that we colour edges between this class and $G \setminus V$ blue. A white ring around a vertex class indicates that we colour edges between this class and $G \setminus V$ red. No ring indicates that such edges retain their original colour. Edges inside V_B are red, edges inside V_R are blue, and edges inside V' retain their original colour.

For all $u \in \{x_B, y_B\} \cup V_B$ and all $v \in G \setminus V$, colour the edge uv blue. For all $u \in \{x_R, y_R\} \cup V_R$ and all $v \in G \setminus V$, colour the edge uv red. It will be

convenient to call the vertices in $\{x_B, y_B\} \cup V_B$ *blue vertices*, and to call the vertices in $\{x_R, y_R\} \cup V_R$ *red vertices*. The recolouring is indicated in Figure 1.

The crucial properties of this recolouring are the following, which are easy to verify from examining Figure 1:

(1) Both G_R and G_B are K_4-free.
(2) Every triangle in G_R contains at least one of $\{x_B\}, \{y_B\}, V_B$, and every triangle in G_B contains at least one of $\{x_R\}, \{y_R\}, V_R$.
(3) The blue vertices $V_B \cup \{x_B, y_B\}$ are connected by only red edges in G, and the red vertices $V_R \cup \{x_R, y_R\}$ are connected by only blue edges in G.

Since G is Ramsey for K_n there must be a monochromatic copy of K_n present in G after this recolouring. We claim that, thanks to the fortuitous properties of our recolouring, this forces a monochromatic K_{n-1} in the original colouring with the required properties.

Let U be the vertex set of the monochromatic K_n present in G after this recolouring. If $|U \cap (V_0 \cup \{x, y\})| \leq 1$ then the lemma follows immediately, since discarding at most one vertex would leave a monochromatic K_{n-1} in the original colouring (as the only edges which are recoloured are incident with $V_0 \cup \{x, y\}$), completely disjoint from $V_0 \cup \{x, y\}$ as required.

We may suppose, therefore, that $|U \cap (V_0 \cup \{x, y\})| \geq 2$. The first case to consider is when $U \subset V$. By Property (1) and since $n \geq 4$, U must contain at least two vertices from one of the classes V_R, V_B, or V'.

Suppose first that $|U \cap V_B| \geq 2$. Then U must form a red K_n, and hence can contain at most one vertex from V_R, and no vertex from V', since all edges between V_B and V' are blue. By similar reasoning, if $|U \cap V_R| \geq 2$, then U must form a blue K_n, and hence cannot use any vertex from $V' \cup V_B$. Therefore, there exists at most one class $V'' \in \{V_R, V_B, V'\}$ such that $|U \cap V''| \geq 2$. Since each such class contains at most $n-3$ vertices, discarding all but one vertex of V'' would force a monochromatic copy of K_4 within V, using at most one vertex from each of V_R, V_B, and V'. This would force a copy of K_4 in either G_R or G_B, which contradicts Property (1).

Assume now that $U \not\subset V$, and suppose that U hosts a red copy of K_n. Since all blue vertices are connected to $G \setminus V$ by blue edges, U cannot contain any blue vertices. Therefore, by Property (2), U uses vertices of at most two nodes in G_R. Furthermore, since the copy is red, $|U \cap V_R| \leq 1$.

If $V' \cap U \neq \emptyset$ then U can use at most one vertex from $V \setminus V'$, and discarding this vertex leaves a monochromatic K_{n-1} in the original colouring, completely disjoint from $V_0 \cup \{x, y\}$, as required.

If $V' \cap U = \emptyset$ then, by Property (2) again, it must use exactly two vertices from $V_R \cup \{x_R, y_R, x, y\}$. Since there are only blue edges between vertices in $V_R \cup \{x_R, y_R\}$, by Property (3), at least one of these two vertices in $U \cap V$ must be x or y. Discarding the other vertex in $U \cap V$ leaves a monochromatic copy of K_{n-1} in the original colouring which intersects V in either x or y, but no other vertices, as required.

The case when U hosts a blue copy of K_n is handled similarly, and the proof is complete. □

The following corollary is immediate when $n \geq 4$; in this weakened form it also holds for $n = 3$. We will not need Corollary 1 in the rest of this paper, but it is a pleasingly simple way to demonstrate the Ramsey stability ethos of Lemma 1.

Corollary 1. *Let $n \geq 3$ and $G \to K_n$. If $V \subset G$ has $|V| \leq 2n-2$ then $G \setminus V \to K_{n-1}$.*

Proof. For $n \geq 4$ this follows immediately from Lemma 1, after expanding V by two arbitrary vertices from $G \setminus V$. For $n = 3$, it suffices to give an explicit colouring of K_4 in a similar fashion, as we do in Figure 2.

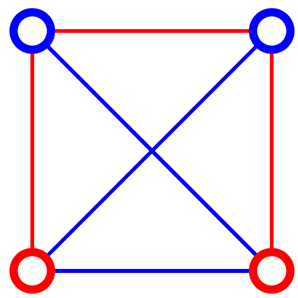


FIGURE 2

Thus, if we recolour the edges adjacent to V as indicated in Figure 2, then any monochromatic K_3 in G must have at least two vertices from $G \setminus V$, and hence $G \setminus V \to K_2$ as required. □

2. PROOF OF THE MAIN RESULT

We recall our goal: to show that K_n is Ramsey equivalent to $K_n + K_{n-1}$ for $n \geq 4$. It is, of course, trivial that if $G \to K_n + K_{n-1}$ then $G \to K_n$. It remains to show that if $G \to K_n$ then $G \to K_n + K_{n-1}$. Our strategy will be to accumulate more and more information about the monochromatic structures present in a colouring of a graph, Ramsey for K_n, without a monochromatic $K_n + K_{n-1}$, until we are eventually able to obtain a contradiction.

Lemma 2. *Let $n \geq 4$. If $G \to K_n$ then, in every colouring of G, there is either a monochromatic $K_n + K_{n-1}$, or a red K_n and a blue K_n.*

Proof. Suppose, without loss of generality, that the edges of G are coloured so that there is a red copy of K_n. Let V_R be the vertex set of this red K_n. As in the proof of Lemma 1, we will recolour some edges of G and use the assumption that $G \to K_n$ to prove the claim.

Suppose first that there is an edge ab of V_R which has the property that every red K_n intersects V_R in at least one vertex besides a and b. In this case, we recolour every other edge of V_R blue, and colour the edges between $V_R \setminus \{a, b\}$ and $G \setminus V_R$ red.

Since $G \to K_n$ there must be a monochromatic K_n in this recoloured G. Suppose first that there is a red K_n. If it uses at least $n-1$ vertices from $G \setminus V_R$ then there is a red K_{n-1} present in $G \setminus V_R$ in the original colouring, and hence a red $K_n + K_{n-1}$. Otherwise, it must use a red edge from V_R. But the only red edge remaining in V_R is ab, and the edges from $\{a, b\}$ to $G \setminus V_R$ retained their original colouring. Therefore, we must have a red K_n in the original colouring that intersects V_R in exactly $\{a, b\}$, which contradicts our choice of ab. Secondly, suppose that there is a

blue K_n in the recoloured G. If it uses any of the new blue edges inside V_R, then it must be contained entirely inside V_R, since the edges from $V_R \setminus \{a,b\}$ to $G \setminus V_R$ are all red. However, this is impossible, since V_R has ab still coloured red. Therefore we must have a blue K_n that uses only edges which were originally blue, and so we have a red K_n and a blue K_n, as required.

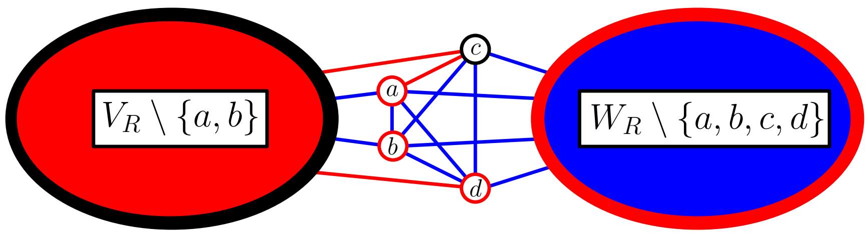


FIGURE 3. The colouring for Lemma 2.

We may now assume that, for every pair $\{a,b\} \subseteq V_R$, there is another red K_n intersecting V_R in only the edge ab. Let W_R be the vertex set of another red K_n such that $|V_R \cap W_R| = 2$, say $V_R \cap W_R = \{a,b\}$, and let c,d be any two vertices in $W_R \setminus V_R$. We recolour (some of) the edges incident to W_R in the following way. An illustration of this colouring can be found in Figure 3.

- For all $w \in W_R \setminus \{a,b,c,d\}$, all $w' \in W_R$ ($w' \neq w$), and all $v \in G \setminus W_R$, we colour the edge ww' blue and the edge wv red (if present in G).
- For all $v \in V_R \setminus \{a,b\}$, we recolour the edges av and bv blue, and the edges cv and dv red (if present in G).
- For all $x \in G \setminus (V_R \cup W_R)$, we colour the edges ax, bx and dx in red (the edge cx retains its original colour).
- Every edge in $\{a,b,c,d\}$ is recoloured blue, except for ac which remains red.

Again, since $G \to K_n$ there must be a monochromatic K_n in this recoloured G.

Suppose first that there is a red K_n, say on vertex set W. If it uses at least $n-1$ vertices from $G \setminus W_R$ then there is a red K_{n-1} present in $G \setminus W_R$ in the original colouring, and hence a red $K_n + K_{n-1}$. Otherwise, it must use a red edge from W_R. But the only red edge remaining in W_R is ac. Then W must be disjoint from $V_R \setminus \{a\}$, since each such ax is blue. Hence, $W \cap (V_R \cup W_R) = \{a,c\}$. But none of the edges inside $W \setminus \{a\}$ were recoloured, and hence $W \setminus \{a\}$ hosts a red K_{n-1} in the original colouring that is vertex disjoint from V_R.

Secondly, suppose that there is a blue K_n in the recoloured G, say on vertex set W. If it uses any of the new blue edges inside $V_R \cup W_R$, then it must be contained entirely inside $V_R \cup W_R$, since the edges from $W_R \setminus \{c\}$ to $G \setminus (V_R \cup W_R)$ are all red. However, $V_R \cup W_R$ does not host a blue K_n in this recolouring. Therefore we must have a blue K_n that uses only edges which were originally blue, and so we have a red K_n and a blue K_n, as required. □

Lemma 3. *Let $n \geq 4$. If $G \to K_n$ then, in any colouring of G, if there is a monochromatic K_{n+1} then there is a monochromatic $K_n + K_{n-1}$.*

Proof. Suppose that G has, say, a red K_{n+1}, on vertex set V_R. By Lemma 2, we may assume that there exists a blue K_n, say on vertex set V_B. Let $V = V_R \cup V_B$,

so that $|V| \leq 2n+1$. We now apply Lemma 1, with $V_0 \subset V$ being any set of $2n-2$ vertices containing V_B. This yields a monochromatic K_{n-1} which intersects the red K_{n+1} in at most one vertex, and the blue K_n not at all, and hence we must have a monochromatic $K_n + K_{n-1}$. □

Lemma 4. *Let $n \geq 4$, and let G be a graph such that $G \to K_n$. Assume that there is a colouring of the edges of G with no monochromatic copy of $K_n + K_{n-1}$. Then, in this colouring, no two monochromatic copies of K_n intersect in exactly two vertices.*

Proof. Suppose otherwise; without loss of generality, we have two red copies of K_n, say on vertex sets V_R and V'_R, such that $|V_R \cap V'_R| = 2$. By Lemma 2 we may further assume that there is a blue K_n, say on vertex set V_B.

Assume first that $V_B \cap V_R \neq \emptyset$. Let $x \in V_R \setminus (V_B \cup V'_R)$ and $y \in V'_R \setminus (V_B \cup V_R)$ (which exist since $n \geq 4$ and since V_B intersects with V_R and V'_R with at most one vertex each). Further, set $V := V_R \cup V'_R \cup V_B$ and $V_0 := (V_R \cup V_B) \setminus \{x\} \subseteq V$. By assumption, $|V| \leq 3n-3$ and $|V_0| \leq 2n-2$. Therefore, by Lemma 1, there is a monochromatic copy of K_{n-1}, say on set W, such that either $W \cap V = \{x\}$, or $W \cap V \subseteq V \setminus (V_0 \cup \{x\})$. In the first case, when $W \cap V = \{x\}$, then W is disjoint from both V_B and V'_R, and hence there is a monochromatic copy of $K_n + K_{n-1}$, a contradiction. Otherwise, W is disjoint from both V_B and V_R, and again, we find a monochromatic copy of $K_n + K_{n-1}$, a contradiction.

We argue similarly if $V_B \cap V'_R \neq \emptyset$, and therefore assume from now on that $V_B \cap (V_R \cup V'_R) = \emptyset$. Let $x, y \in V_R \setminus V'_R$ and $z \in V'_R \setminus V_R$ be some arbitrarily chosen vertices. We again apply Lemma 1, with $V := V_B \cup V_R \cup W$, where $W = V'_R \setminus (V_R \cup \{z\})$, and $V_0 := (V_R \cup V_B) \setminus \{x, y\}$. It is clear that $|V| \leq 3n-3$ and $|V_0| = 2n-2$, as required.

Suppose that there is a monochromatic copy of K_{n-1} which intersects V in only vertices of W. In particular, it is vertex-disjoint from $V_B \cup V_R$, and hence it creates a monochromatic $K_n + K_{n-1}$, which is a contradiction.

It follows that there exists a monochromatic copy of K_{n-1} which intersects V in either x or y, but no other vertices. Since it is disjoint from V_B, we may assume that it is red. If this red K_{n-1} does not use z, however, then together with V'_R we have a red $K_n + K_{n-1}$, which is a contradiction. Therefore, either xz or yz is red. Since x and y were an arbitrary choice of two vertices from $V_R \setminus V'_R$, it follows that all but at most one vertex of V_R is connected to z by a red edge.

That is, $V_R \cup \{z\}$ hosts two red copies of K_n that intersect in $n-1$ vertices. Note that if $V_R \cup \{z\}$ forms in fact a red copy of K_{n+1}, then we are done by Lemma 3. Therefore, to finish the argument, let $x \in V_R \setminus V'_R$ such that the edge xz is blue or not present in G. As noted, there is at most one such x. We apply Lemma 1 yet again to reach a contradiction. Let $y \in V_R \setminus (V'_R \cup \{x\})$, set $V_0 := (V_R \cup V_B) \setminus \{x, y\}$ and $V := (V_R \cup V'_R \cup V_B) \setminus \{x\}$. Then $|V| = 3n-3$ and $|V_0| = 2n-2$. By Lemma 1, there exists a monochromatic copy of K_{n-1}, say on vertex set W, such that either $W \cap V = \{y\}$, $W \cap V = \{z\}$, or $W \cap V \subseteq V \setminus (V_0 \cup \{y, z\})$. If $W \cap V = \{y\}$, then W is disjoint from $V'_R \cup V_B$ and hence forms a monochromatic copy of $K_n + K_{n-1}$ in the original colouring, a contradiction. If $W \cap V = \{z\}$, then W is disjoint from V_B, and hence we may assume that it is red. But then, W is either disjoint from V_R and forms a red copy of $K_n + K_{n-1}$, or $x \in W$, and hence the edge zx is red, a contradiction. Finally, if $W \cap V \subseteq V \setminus (V_0 \cup \{y, z\})$, then W together with $V_0 \cup \{y, z\}$ forms a monochromatic copy of $K_n + K_{n-1}$. □

We will now conclude the proof of the main result.

Proof of Theorem 1. Let $n \geq 4$, and let G be a graph such that $G \to K_n$. Assume that there exists a colouring of the edges of G without a monochromatic copy of $K_n + K_{n-1}$. By Lemma 2, we can assume that there are two (not necessarily disjoint) sets V_R and V_B of vertices such that $G[V_R]$ and $G[V_B]$ form a red and a blue copy of K_n, respectively.

By assumption, any other red (blue) copy of K_n intersects V_R (V_B) in at least two vertices; in fact, by Lemma 4, any other red (blue) copy of K_n intersects V_R (V_B) in at least three vertices. That is, every set $W_R \subset V_R$ of size $|W_R| = n-2$ meets every red copy of K_n in at least one vertex, and every set $W_B \subset V_B$ of size $|W_B| = n-2$ meets every blue copy of K_n in at least one vertex.

If $V_R \cap V_B = \emptyset$, fix two arbitrary subsets $W_R \subset V_R$ and $W_B \subset V_B$, both of size $|W_R| = |W_B| = n-2$. If $V_R \cap V_B \neq \emptyset$, let $W_B \subseteq V_B$ be a set of size $n-2$ such that $V_R \cap V_B \subseteq W_B$, and let $W_R \subseteq V_R$ be a subset of size $n-3$ such that $W_R \cap V_B = \emptyset$ (note that $|V_R \cap V_B| = 1$). In both cases, the sets $W_R \subset V_R$ and $W_B \subset V_B$ are disjoint and, by the above discussion, any monochromatic copy of K_n meets $W_R \cup W_B$ in at least one vertex.

We now recolour the graph and show that the resulting colouring does not contain a monochromatic copy of K_n. We may assume, without loss of generality, that all edges in $V_R \cup V_B$ are present, since losing edges will only help prevent a monochromatic K_n occurring. Let $\{x_R, y_R\} = V_R \setminus (W_R \cup W_B)$ and $\{x_B, y_B\} = V_B \setminus (W_R \cup W_B)$.

- If $n = 4$ and $V_R \cap V_B \neq \emptyset$ (i.e. $|W_R| = 1$), colour one edge between W_R and W_B red, and the other one blue. Otherwise, colour the edges between W_R and W_B so that for every $v \in W_R$ there are $w_r, w_b \in W_B$ such that vw_r is red and vw_b is blue, and for every $v \in W_B$ there are $w_r, w_b \in W_R$ such that vw_r is red and vw_b is blue.[2]
- For all $x \in W_R$, $y \in V_R$, and $z \notin V_R \cup W_B$, colour the edge xy blue and colour the edge xz red.
- For all $x \in W_B$, $y \in V_B$, and $z \notin W_R \cup V_B$, colour the edge xy red and colour the edge xz blue.

This recolouring is illustrated in Figure 4 (where we label as black those edges which retain their original colouring).

Note that we only recolour edges incident to $W_R \cup W_B$. Therefore, by our choice of $W_R \cup W_B$, any monochromatic copy of K_n (after recolouring the edges) must meet $W_R \cup W_B$ in at least one vertex.

Suppose now that a red K_n exists and uses vertices from W_R but not W_B. Then it must use just one vertex from W_R and $n-1$ from $G \setminus (V_R \cup W_B)$, and hence we have a red $K_n + K_{n-1}$ in the original colouring. If a blue K_n exists and uses vertices from W_R but not W_B, then it cannot use any vertices from $\{x_B, y_B\}$ or $G \setminus (V_R \cup V_B)$, and can only use at most one vertex from $\{x_R, y_R\}$ (since the edge $x_R y_R$ remains red as in the original colouring). But this contradicts the fact that $|W_R| \leq n-2$.

Similarly, we can rule out the case that a monochromatic copy of K_n uses vertices from W_B but not W_R. Therefore, if there is a monochromatic copy of K_n after recolouring the edges, then it must use vertices from both W_R and W_B. Assume

[2] This is clearly possible if $|W_R|, |W_B| \geq 2$, i.e. if $V_R \cap V_B = \emptyset$ or $n \geq 5$.

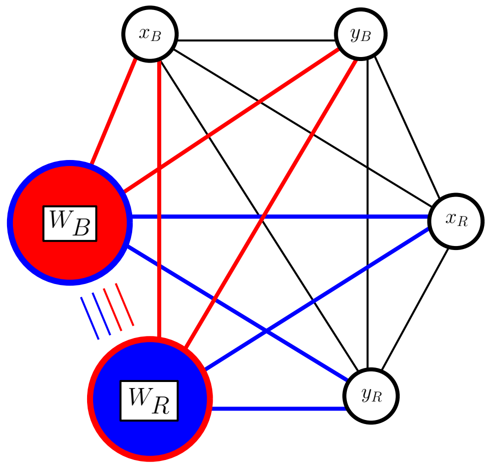


FIGURE 4. The colouring for the proof of Theorem 1.

first that this copy is red. Since all vertices in W_B are connected to $G \setminus (V_R \cup V_B)$ via blue edges, the red copy of K_n must lie entirely inside $V_R \cup V_B$. But then, it can use at most one vertex from W_R and at most one of $\{x_B, y_B\}$. The remaining $n-2$ vertices must come from W_B, so we must use all vertices from W_B. However, in the case $V_R \cap V_B = \emptyset$ or $n \geq 5$, every vertex in W_R sees at least one vertex of W_B in blue. In the case $n = 4$, $|W_R| = 1$ and $|W_B| = 2$, the two edges between W_R and W_B are of opposite colour, and hence, at most one vertex of W_B can contribute to a red K_4.

A similar argument shows that we do not find a blue copy of K_n using vertices from both W_R and W_B. We have therefore constructed a colouring of G which has no monochromatic K_n, contradicting the original Ramsey property of G and concluding the proof. □

3. Ramsey equivalence of K_3

In this section we give a proof of Theorem 2, a result of Szabó and Bodkin [2]. We need to show that, if $G \to K_3$ and $G \nrightarrow K_3 + K_2$, then $K_6 \subset G$.

Proof of Theorem 2. Let G be a graph which is Ramsey for K_3 and not Ramsey for $K_3 + K_2$, and fix some colouring of G with no monochromatic $K_3 + K_2$. We first show that G must possess both a red K_3 and a blue K_3.

Without loss of generality, there is a red K_3, say on vertex set $V_R = \{x_R, y_R, z_R\}$. We now recolour the edges $x_R y_R$ and $x_R z_R$ blue, and colour all the edges from x_R to $G \setminus V_R$ red. It is now straightforward that a blue copy of K_3 must be a blue copy in the original colouring, and that a red copy of K_3 forces either a monochromatic copy of $K_3 + K_2$ in the original colouring, or it uses the edge $y_R z_R$ and single new vertex, say v_R. In this case, we recolour once again in the following way, as indicated in Figure 5.

We colour the three-edge path (z_R, x_R, v_R, y_R) red, and the complement in $V_r \cup \{v_R\}$ blue. Furthermore, we colour all edges between $\{z_R, y_R\}$ and $G - (V_r \cup \{v_R\})$ red. As before, if there is now a blue K_3, then it cannot use either of the vertices

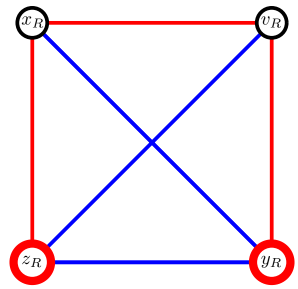


FIGURE 5

y_R or z_R, and hence it must have been already present in the original colouring of G. Otherwise, a red K_3 must use exactly two vertices from $\{x_R, v_R, y_R, z_R\}$. In particular, we have a red K_2 that is either disjoint from $\{x_R, y_R, z_R\}$ or $\{v_R, y_R, z_R\}$, and hence a red $K_3 + K_2$ in the original colouring.

We have shown that there must be, in our coloured graph G, a red K_3, say on V_R, and a blue K_3, say on V_B. We now show that we can assume that V_R and V_B are disjoint.

Suppose that our original choices are not, so that $|V_R \cup V_B| = 5$. Suppose $V_R \cap V_B = \{x\}$ and $V_R = \{x, y_R, z_R\}$ and $V_B = \{x, y_B, z_B\}$. Clearly, any edges between $\{y_R, z_R\}$ and $G \setminus (V_R \cup V_B)$ must be red. If their neighbourhoods intersect in $G \setminus (V_R \cup V_B)$ we have found another red K_3, entirely disjoint from V_B, and we may proceed. Otherwise, we may assume that the neighbourhoods of y_R and z_R in $G \setminus (V_R \cup V_B)$ are disjoint. Similarly, we can assume that the neighbourhoods of y_B and z_B in $G \setminus (V_R \cup V_B)$ are disjoint. We now colour the edges incident to $V_R \cup V_B$ as indicated in Figure 6. Since $G \to K_3$, there must be a monochromatic copy of K_3 after recolouring. Furthermore, it must intersect $V_R \cup V_B$ in exactly two vertices, since the original colouring would contain a monochromatic $K_3 + K_2$ otherwise. If it is a red K_3, say, then it must therefore use y_R, z_R, and a single vertex from $G \setminus (V_R \cup V_B)$, which contradicts the fact that their neighbourhoods are disjoint as discussed above, and we argue similarly if we have found a blue K_3.

We may therefore assume that we have produced two disjoint sets, V_R and V_B, each of which spans a red and blue K_3 respectively.

Suppose first that there are two vertex-disjoint edges missing from $V_R \cup V_B$. We then recolour the edges incident to $V_R \cup V_B$ as in Figure 7 (where, as usual, a red (blue) vertex represents the fact that the edges between that vertex and $G \setminus (V_R \cup V_B)$ are coloured red (blue)). It is easy to check that this colouring of $V_R \cup V_B$ contains no monochromatic K_3. Moreover, there are no blue edges between blue vertices, and, vice versa, no red edges between red vertices. It follows that a monochromatic copy of K_3 in this recoloured G must use at least two vertices from $G \setminus (V_R \cup V_B)$, which would create a monochromatic $K_3 + K_2$ in the original colouring of G, a contradiction.

We may suppose, therefore, that there is a vertex, without loss of generality say $x_R \in V_R$, such that every missing edge in $V_R \cup V_B$ is adjacent to x_R. Furthermore, we may suppose that at least one edge is missing, or else we have a K_6 in G as required. Let $x_R x_B$ be some missing edge, where $x_B \in V_B$.

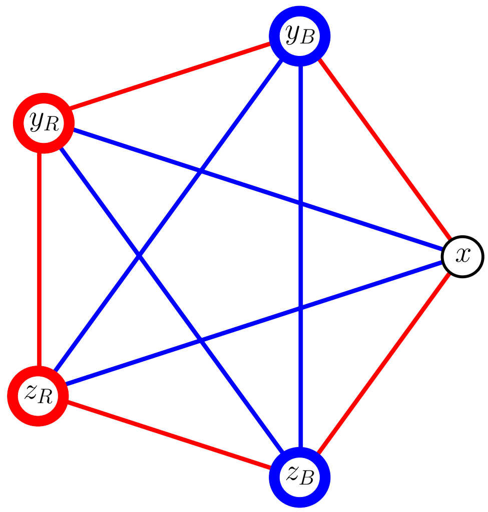


FIGURE 6. The recolouring when $V_R \cap V_B = \{x\}$. The edges between $\{y_R, z_R\}$ and $G - (V_R \cup V_B)$ are red, the edges between $\{y_B, z_B\}$ and $G - (V_R \cup V_B)$ are blue.

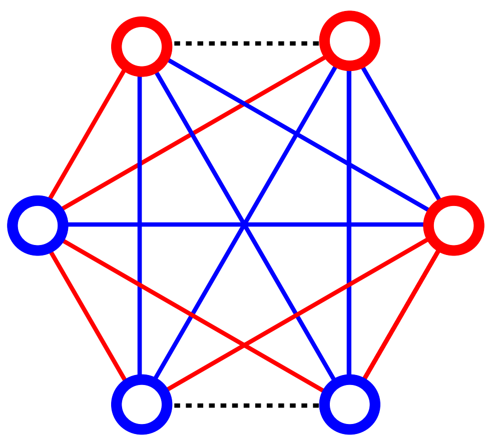


FIGURE 7

Assume first that there is a vertex, say w, in $G \setminus (V_R \cup V_B)$ that has at least five neighbours in $V_R \cup V_B$. If it is adjacent to every vertex of $(V_R \cup V_B) \setminus \{x_R\}$ then this creates a K_6, as required. Hence, we can assume that wx_R is an edge in G. Furthermore, all edges between w and V_R (if present in G) must be red, and all edges between w and V_B must be blue (as otherwise they create a monochromatic copy of $K_3 + K_2$ in the original colouring).

Suppose that w is adjacent to every vertex of V_R and to two vertices of V_B, say a and b, and that the edge wc is missing. If either of the edges x_Ra or x_Rb is missing, then by considering $\{w, x_R, y_R\} \cup V_B$ we have a similar situation as above – namely, disjoint vertex sets of a red and a blue copy of K_3 with two vertex disjoint

edges missing, and we are done. Otherwise, we have a K_6 in $\{w, x_R, y_R, z_R, a, b\}$. Suppose now that w is adjacent to every vertex of V_B and x_R and some other vertex of V_R, say a, and the edge wb is missing, where $b \in V_R$. As above, we are now done by considering $V_R \cup \{w, x_B, y_B\}$, since wb and $x_B x_R$ are two independent edges missing.

For the remainder of the argument, we may therefore assume that every vertex of $G \setminus (V_R \cup V_B)$ has at most four neighbours in $V_R \cup V_B$. We now describe a recolouring of the edges incident to $V_R \cup V_B$ such that there is no monochromatic K_3 that uses at least two vertices from $V_R \cup V_B$. Recolour the interior edges of $V_R \cup V_B$ as in Figure 8. Let now $w \in G \setminus (V_R \cup V_B)$ and let $N_w \subseteq V_R \cup V_B$ be any

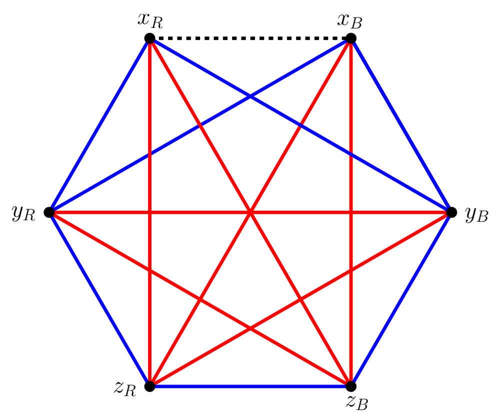


FIGURE 8

set of four vertices containing $N(w) \cap (V_R \cup V_B)$. Then, either (1), $\{x_R, x_B\} \not\subseteq N_w$ and we see a red copy of the three-edge path P_3 and a blue copy of P_3 in N_w, or (2), $\{x_R, x_B\} \subseteq N_w$, say $N_w = \{a, b, x_R, x_B\}$, for some a, b, and we see a monochromatic copy of C_4 or a monochromatic star $K_{1,3}$ with a being the centre of the star.

In case (1), say (a, b, c, d) forms the red P_3 in N_w, then we colour the edges wb and wc blue, and the edges wa and wd red (if present in G). In case (2), we colour the edge wa, wx_R and wx_B the opposite colour of ax_R and wb the same colour as ax_R.

Note first that the colouring of $V_R \cup V_B$ does not contain a monochromatic triangle. Furthermore, it is evident that we do not create a monochromatic triangle on vertices w, x, y with $x, y \in V_R \cup V_B$ and $w \notin V_R \cup V_B$, since no such w sees both vertices of a red edge in red nor both vertices of a blue edge in blue.

However, since $G \to K_3$, there must be an edge vw with $v, w \notin V_R \cup V_B$, which creates a monochromatic $K_3 + K_2$ in the original colouring, a contradiction. $\square$

4. FURTHER REMARKS

We have shown that K_n and $K_n + K_{n-1}$ are Ramsey equivalent for $n \geq 4$. Furthermore, we have seen that K_6 is the only obstruction to the Ramsey equivalence of K_3 and $K_3 + K_2$, i.e. any graph G that satisfies $G \to K_3$ and $G \nrightarrow K_3 + K_2$ must contain K_6 as a subgraph.

The only pairs of graphs (H_1, H_2) known to be Ramsey equivalent are of the form $H_1 \cong K_n$ and $H_2 \cong K_n + H_3$, where H_3 is a graph of clique number less than

n. Furthermore, it is known ([6] and [8]) that the only connected graph that is Ramsey equivalent to K_n is the clique K_n itself.

It is an open question, first posed in [6], whether there are two connected non-isomorphic graphs H_1 and H_2 that are Ramsey equivalent. It follows from [8] that, if such a pair exist, they must have the same clique number. In [1] it is shown that they must also have the same chromatic number, under the assumption that one of the two graphs satisfies an additional property, called *clique-splittability*.

To tackle problems on Ramsey equivalence, a weaker concept was proposed by Szabó [9]. We will first introduce some necessary notation. We say that G is Ramsey minimal for H if G is Ramsey for H and no proper subgraph of G is Ramsey for H. Denote by $\mathcal{M}(H)$ the set of all graphs which are Ramsey minimal for H, and by $\mathcal{R}(H)$ the set all graphs which are Ramsey for H. Finally, let $\mathcal{D}(H_1, H_2) := (\mathcal{M}(H_1) \setminus \mathcal{R}(H_2)) \cup (\mathcal{M}(H_2) \setminus \mathcal{R}(H_1))$ be the class of graphs G that are Ramsey minimal for H_1, but which are not Ramsey for H_2, or vice versa. Equivalently, $\mathcal{D}(H_1, H_2)$ is the set of minimal obstructions to the Ramsey equivalence of H_1 and H_2.

In particular, H_1 and H_2 are Ramsey equivalent if and only if $\mathcal{D}(H_1, H_2) = \emptyset$. We say that H_1 and H_2 are Ramsey close, denoted by $H_1 \sim_c H_2$, if $\mathcal{D}(H_1, H_2)$ is finite. We stress that this is not an equivalence relation: reflexivity and symmetry are trivial, but transitivity does not hold, since every graph containing at least one edge is close to K_2.

Two graphs may be Ramsey close in a rather trivial sense if $\mathcal{M}(H_1)$ and $\mathcal{M}(H_2)$ are both finite, or if $H_2 \subset H_1$ and $\mathcal{M}(H_2)$ is finite. Graphs such that $\mathcal{M}(H)$ is finite are known as Ramsey-finite graphs. The class of Ramsey-finite graphs has been studied quite intensively; see, for example, [3] for some results and further references. In particular, it has been shown that the only Ramsey-finite graphs are disjoint unions of stars.

If one wishes to prove that two graphs are Ramsey equivalent, a possible first step is to show that the two graphs are Ramsey close. Szabó [9] has posed the following weaker version of the open problem mentioned earlier.

Question 1. *Is there a pair of non-isomorphic, Ramsey-infinite, connected graphs which are Ramsey close?*

We suspect that the answer to Question 1 is negative, even with this weakening of the notion of Ramsey equivalence.

Nešetřil and Rödl [7] proved that if $\omega(H) \geq 3$ then there exist infinitely many Ramsey-minimal graphs $G \in \mathcal{M}(H)$ such that $\omega(H) = \omega(G)$. In particular, it follows that if $\omega(G_1) \geq 3$ and $\omega(G_2) \geq 3$, and $G_1 \sim_c G_2$, then $\omega(G_1) = \omega(G_2)$.

Theorem 2 states that, although K_3 and $K_3 + K_2$ are not Ramsey equivalent, they are Ramsey close. Indeed, the only graph G that is Ramsey minimal for K_3 and satisfies $G \nrightarrow K_3 + K_2$ is K_6 itself. This is the only example of a pair of Ramsey-infinite graphs which are Ramsey close but not Ramsey equivalent that we know of. In this case, $|\mathcal{D}(K_3, K_3 + K_2)| = 1$. We pose the following.

Question 2. *For any integer $k \geq 2$, is there a pair of Ramsey-infinite graphs H_1 and H_2 such that $|\mathcal{D}(H_1, H_2)| = k$?*

An affirmative answer, which we believe to exist, would in particular imply the following conjecture.

Conjecture 1. *There are infinitely many pairs of Ramsey-infinite graphs which are Ramsey close but not Ramsey equivalent.*

We close this paper with the following simpler question, which one may be able to answer negatively by adapting the methods of [6].

Question 3. *Are K_n and $K_n + K_n$ Ramsey close?*

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